Problem Set 8 Solutions: Continuous Distributions Solutions
CS&SS Math Camp 2020

1. While driving to the Sounders' game, you stop at a stoplight at 8:47:00. The time you will have to wait there follows a continuous uniform distribution from 8:47:00 to 8:50:00.

   (a) What is the probability that you will have to wait at least a minute?
   Let $X \sim Uniform(0, 3)$, 3 minutes being the length of time between 8:47 and 8:50. Then,

   $$f(x) = \frac{1}{3-0} = \frac{1}{3}.$$

   So,

   $$P(X \geq 1) = P(1 \leq X \leq 3) = \int_1^3 \frac{1}{3} dx = \frac{1}{3} x|_1^3 = \frac{1}{3}(3-1) = \frac{2}{3}$$

   (b) What is the probability that you will wait more than 2 minutes?

   $$P(X > 2) = P(2 < X \leq 3) = \int_2^3 \frac{1}{3} dx = \frac{1}{3} x|_2^3 = \frac{1}{3}(3-2) = \frac{1}{3}$$

   (c) What is the probability that you will wait less than 4 minutes?

   $$P(X \leq 4) = P(0 \leq X \leq 3) = \int_0^3 \frac{1}{3} dx = \frac{1}{3} x|_0^3 = 1 - 0 = 1$$

   NOTE: we are told the longest you can wait is 3 minutes.

   (d) What is your expected wait time?

   $$E[X] = \frac{a+b}{2} = \frac{0+3}{2} = 1.5\text{min}$$

   OR

   $$E[X] = \int_0^3 x \cdot \frac{1}{3} dx = \frac{1}{3} \cdot \frac{x^2}{2}|_0^3 = \frac{1}{6}\left(3^2 - 0^2\right) = \frac{9}{6} = \frac{3}{2} = 1.5\text{min}$$

2. Suppose we ask 30 people whether they believe Stefan Frei is doing a good job as Seattle Sounders keeper. 19 people said they agree, 6 said they were neutral, and 5 said they disagree. We believe the city is evenly split between the three viewpoints. Compute the chi-square test statistic to determine whether our belief is reasonable.

$$\chi^2 = \sum \frac{(O-E)^2}{E} = \frac{(19-10)^2}{10} + \frac{(6-10)^2}{10} + \frac{(5-10)^2}{10} \approx 12.2$$

The degrees of freedom (d.o.f) is $3 - 1 = 2$. We know the expected value of a chi-square distribution is the d.o.f. and the variances is the d.o.f. squared. Thus, we are asking if observing a value of 2.6 is rare from a distribution with mean 2 and variance 4. The answer is no, it is not unusual to observe such a value so we would say there is not enough evidence to suggest our belief that the population is evenly split between the three viewpoints is unreasonable. To formally test this we can calculate the p-value in R:

```
> 1-pchisq(12.2)
[1] 0.002242868
```

This suggests that the observation is very unlikely if we assume the city is evenly split. We will reject the null hypothesis that the opinion is split evenly. (We already knew Frei was beloved though.)

3. Suppose that the number of hours graduate students sleep is normally distributed with mean 6 and standard deviation 1. (HINT: use the 68-95-99 Rule or `pnorm` in `R`.)

   (a) What is the probability that a randomly chosen grad student slept 8 or more hours last night?

   $$P(X > 8) = P\left(\frac{X-\mu}{\sigma} > \frac{8-6}{1}\right) = P(Z > 2) = 0.025$$

   ```
   > 1-pnorm(2)
   [1] 0.02275013
   ```

   (b) What is the probability that a randomly chosen grad student slept less than 4 hours last night?

   $$P(X < 4) = P\left(\frac{X-\mu}{\sigma} < \frac{4-6}{1}\right) P(Z < -2) = 0.025$$

   ```
   > pnorm(-2)
   [1] 0.02275013
   ```

   (c) What is the probability that a randomly chosen grad student slept between 5 and 7 hours last night?

   $$P(5 < X < 7) = P\left(\frac{5-6}{1} < \frac{X-\mu}{\sigma} < \frac{7-6}{1}\right) = P(-1 < Z < 1) = P(Z < 1) - P(Z < -1) = 0.68$$

   ```
   > pnorm(1)-pnorm(-1)
   [1] 0.6826895
   ```