Problem Set 2: Matrix Algebra Solutions
CS&SS Math Camp 2020

$A = \begin{pmatrix} 2 & 7 \\ 1 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 6 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 7 & 1 \\ 2 & 1 & 6 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}$ $E = \begin{pmatrix} 1 & 4 \\ 9 & 3 \\ 7 & 2 \end{pmatrix}$ Compute the following (if you can):

- $B^t = \begin{pmatrix} 1 & 4 \\ 3 & 2 \\ 7 & 6 \end{pmatrix}$

- $A + D = \begin{pmatrix} 5 & 11 \\ 6 & 9 \end{pmatrix}$

- $C_{3\times3} + D_{2\times2}$
  Cannot compute $C + D$ since C and D have different dimensions.

- D^{-1}
  First compute the determinant of D to see whether or not it can be inverted.

  $$D(D) = 3 \cdot 1 - 4 \cdot 5 = 3 - 20 = -17$$

  $D(D) \neq 0$ so the matrix is invertible.

  $$D^{-1} = \frac{1}{-17} \begin{pmatrix} 1 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{17} & \frac{4}{17} \\ \frac{5}{17} & -\frac{3}{17} \end{pmatrix}$$

- $C \cdot B$
  Cannot compute $C_{3\times\mathbf{3}} \cdot B_{\mathbf{2}\times3}$ because the number of columns in C is NOT the same as the number of rows of B, i.e. the "inside" dimensions do not match.

- $B \cdot E = \begin{pmatrix} 77 & 27 \\ 64 & 34 \end{pmatrix}$

- $E \cdot B = \begin{pmatrix} 17 & 11 & 31 \\ 21 & 33 & 81 \\ 15 & 25 & 61 \end{pmatrix}$

- $A \cdot E$
  Cannot compute $A \cdot E$ because the number of columns in A is NOT the same as the number of rows of E.

- $A \cdot E^t = \begin{pmatrix} 30 & 39 & 28 \\ 33 & 33 & 23 \end{pmatrix}$

Find the determinants. Do the inverses exist?

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 \\ 5 & 10 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 7 \\ 9 & 21 \end{pmatrix}$$

$$D(A) = 6 \times 4 - 3 \times 1 = 21$$
$$D(B) = 1 \times 10 - 5 \times 3 = -5$$
$$D(C) = 3 \times 21 - 9 \times 7 = 0$$

Matrices A and B are **nonsingular** and the inverses exist. Matrix C is **singular** and the inverse does not exist.

Solve the following systems of equations using matrices.

- $4x + 9y = 31$, $2x + 3y = 11$

$$A = \begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix} \quad z = \begin{pmatrix} x \\ y \end{pmatrix} \quad w = \begin{pmatrix} 31 \\ 11 \end{pmatrix}$$

$$D(A) = 12 - 18 = -6$$
$$A^{-1} = \frac{1}{-6} \begin{pmatrix} 3 & -9 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{6} & \frac{9}{6} \\ \frac{2}{6} & -\frac{4}{6} \end{pmatrix}$$

$$
\begin{aligned}
z =& A^{-1}w \\
=& \begin{pmatrix} -\frac{3}{6} & \frac{9}{6} \\ \frac{2}{6} & -\frac{4}{6} \end{pmatrix} \cdot \begin{pmatrix} 31 \\ 11 \end{pmatrix} \\
=& \begin{pmatrix} -\frac{3}{6} \times 31 + \frac{9}{6} \times 11 \\ \frac{2}{6} \times 31 + -\frac{4}{6} \times 11 \end{pmatrix}_{2\times 1} \\
=& \begin{pmatrix} 13 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}
$$

- $5y - 2x = 10$, $4x - 3y = 6$

$$
A = \begin{pmatrix} -2 & 5 \\ 4 & -3 \end{pmatrix} \qquad z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad w = \begin{pmatrix} 10 \\ 6 \end{pmatrix}
$$

$$
D(A) = 6 - 20 = -14
$$

$$
A^{-1} = \frac{1}{-14} \begin{pmatrix} -3 & -5 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{14} & \frac{5}{14} \\ \frac{4}{14} & \frac{2}{14} \end{pmatrix}
$$

$$
\begin{aligned}
z =& A^{-1}w \\
=& \begin{pmatrix} \frac{3}{14} & \frac{5}{14} \\ \frac{4}{14} & \frac{2}{14} \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 6 \end{pmatrix} \\
=& \begin{pmatrix} \frac{3}{14} \times 10 + \frac{5}{14} \times 6 \\ \frac{4}{14} \times 10 + \frac{2}{14} \times 6 \end{pmatrix} \\
=& \begin{pmatrix} \frac{30}{7} \\ \frac{52}{7} \end{pmatrix}
\end{aligned}
$$

You ask 5 people 4 questions about their high school academics and note their answers (i.e. number of math classes in high school, number of extra curricular activities). You also write down their college GPA. You want to find the relationship between the high school academics questions (independent variables, x) and their college GPA (dependent variable, y). i.e. you would like to find the β matrix in $y = X\beta$.

Person 1: $x_1 = 8$, $x_2 = 12$, $x_3 = 2$, $x_4 = 16$; $y = 3.6$

Person 2: $x_1 = 9$, $x_2 = 7$, $x_3 = 3$, $x_4 = 18$; $y = 3.3$

Person 3: $x_1 = 5$, $x_2 = 13$, $x_3 = 1$, $x_4 = 15$; $y = 3.9$

Person 4: $x_1 = 4$, $x_2 = 9$, $x_3 = 2$, $x_4 = 20$; $y = 3.7$

Person 5: $x_1 = 7$, $x_2 = 11$, $x_3 = 3$, $x_4 = 21$; $y = 3.8$

Write down y, X. (We would use a computer to solve for β).

$$y = \begin{pmatrix} 3.6 \\ 3.3 \\ 3.9 \\ 3.7 \\ 3.8 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 8 & 12 & 2 & 16 \\ 1 & 9 & 7 & 3 & 18 \\ 1 & 5 & 13 & 1 & 15 \\ 1 & 4 & 9 & 2 & 20 \\ 1 & 7 & 11 & 3 & 21 \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

EXTRA CREDIT: use a computer to solve for β.

```
### in R ###
y<-c(3.6,3.3,3.9,3.7,3.8)
x1<-c(1,8,12,2,16)
x2<-c(1,9,7,3,18)
x3<-c(1,5,13,1,15)
x4<-c(1,4,9,2,20)
x5<-c(1,7,11,3,21)
X<-as.matrix(rbind(x1,x2,x3,x4,x5))
solve(t(X) %*% X ) %*% t(X) %*%y
           [,1]
[1,]  7.1166667
[2,] -0.4083333
[3,]  0.1333333
[4,]  1.3416667
[5,] -0.2833333
```